

LIIOUVILLE TYPE THEOREMS FOR TWO MIXED BOUNDARY VALUE PROBLEMS WITH GENERAL NONLINEARITIES

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Abstract: In this paper, we study the nonexistence of positive solutions for the following two mixed boundary value problems. The first problem is the mixed nonlinear-Neumann boundary value problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}_+^N, \\ \frac{\partial u}{\partial \nu} = g(u) & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_0 \end{cases}$$

and the second is the nonlinear-Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}_+^N, \\ \frac{\partial u}{\partial \nu} = g(u) & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_0, \end{cases}$$

where $\mathbb{R}_+^N = \{x \in \mathbb{R}^N : x_N > 0\}$, $\Gamma_1 = \{x \in \mathbb{R}^N : x_N = 0, x_1 < 0\}$ and $\Gamma_0 = \{x \in \mathbb{R}^N : x_N = 0, x_1 > 0\}$. We will prove that these problems possess no positive solution under some assumptions on the nonlinear terms. The main technique we use is the moving plane method in an integral form.

keywords: Liouville type theorem, Moving plane method, Maximum principle, Mixed boundary value.

1. Introduction

In this paper, we study the nonexistence of positive solutions for the following two mixed boundary value problems with general nonlinearities. The first one is the nonlinear-Neumann boundary value problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}_+^N, \\ \frac{\partial u}{\partial \nu} = g(u) & \text{on } \Gamma_1, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_0 \end{cases} \quad (1.1)$$

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and the second one is the nonlinear-Dirichlet problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}_+^N, \\ \frac{\partial u}{\partial \nu} = g(u) & \text{on } \Gamma_1, \\ u = 0 & \text{on } \Gamma_0, \end{cases} \quad (1.2)$$

where $\mathbb{R}_+^N = \{x \in \mathbb{R}^N : x_N > 0\}$, $N \geq 3$, $\Gamma_1 = \{x \in \mathbb{R}^N : x_N = 0, x_1 < 0\}$ and $\Gamma_0 = \{x \in \mathbb{R}^N : x_N = 0, x_1 > 0\}$. In the following, we assume that f, g are continuous functions.

Liouville type theorems are close related to the existence results and prior estimates for elliptic equations. More precisely, after blowing up, elliptic equation in bounded domain turns to be an equation in \mathbb{R}^N or \mathbb{R}_+^N . Using the respective Liouville theorem, we get a contradiction, so the prior estimate is proved. Then we can use the topological method to prove the existence results. For more details, we refer to [9][11] and etc.

In the past few decades, there are plenty of works on the Liouville type theorems for elliptic equations and elliptic systems. The first result is [10], in which the authors studied the nonexistence results for the following elliptic equation

$$-\Delta u = u^p \quad \text{in } \mathbb{R}^N, \quad u \geq 0. \quad (1.3)$$

The authors proved, among other things, that the only solution for problem (1.3) is $u \equiv 0$ provided $0 < p < \frac{N+2}{N-2}$. This result is optimal in the sense that for any $p \geq \frac{N+2}{N-2}$, there are infinitely many positive solutions to (1.3). Thus the Sobolev exponent $\frac{N+2}{N-2}$ is the dividing exponent between existence and nonexistence of positive solutions. Nonexistence result for problem in half space was also obtained in [11]. Later, W.Chen and C.Li proved similar results by using the moving plane method in [4]. By using the moving plane method, the authors proved the solution is symmetric in every direction and with respect every point, hence the solution must be $u \equiv 0$. Recently, in an interesting paper [6], L.Damascelli and F.Gladiali studied the nonexistence result of positive solution for the following nonlinear problem with general nonlinearity

$$-\Delta u = f(u) \quad \text{in } \mathbb{R}^N, \quad u \geq 0. \quad (1.4)$$

If f is assumed to be increasing and subcritical, then the authors proved the only solution for problem (1.4) is $u \equiv 0$. The main tool they used is the method of moving planes. We note that f is only assumed to be continuous in this paper. So we can not conclude that the weak solutions of problem (1.4) are of C^2 class. In developing the method of moving plane, the authors in [6] used the technique based on integral inequalities, an idea originally due to S.Terracini's work [22] and [23]. After the work of [6], a lot of works concern the Liouville type theorems for elliptic equation with general nonlinearity, see [12][13][14][15][24][25][26][27]. For Liouville theorem on nonlinear elliptic systems, we refer to [8][16][18][19][20][21] and etc..

In this paper, we are concerned with the nonexistence of positive solution for the two mixed boundary value problems. Mixed boundary problems were widely studied in the

past few decades. For example, E.Colorado and I.Peral studied the existence results in [5]. The concentration behaviors of singularly perturbed mixed boundary problems were studied in [1][2][7] and the references therein. Nonexistence results for Neumann-Dirichlet mixed boundary problem have also been obtained in the past, see [3] and [6]. However, up to our knowledge, it seems to no result for nonlinear-Neumann and nonlinear-Dirichlet mixed boundary value problems and this is the main purpose of this paper.

First, we need to define weak solution for problem (1.1) and problem (1.2). Let $E = W_{loc}^{1,2}(\mathbb{R}_+^N) \cap C^0(\overline{\mathbb{R}_+^N})$, we say that u is a weak solution of problem (1.1), if $u \in E$ and satisfies

$$\int_{\mathbb{R}^N} \nabla u \nabla \varphi \, dx = \int_{\mathbb{R}^N} f(u) \varphi \, dx + \int_{\Gamma_1} g(u) \varphi \, dx'$$

for any $\varphi \in C_c^1(\overline{\mathbb{R}_+^N})$. Similarly, we denote by $W = \{\varphi \in C_c^1(\overline{\mathbb{R}_+^N}), \text{supp}\{\varphi\} \subset A\}$ with $A = \mathbb{R}_+^N \cup \Gamma_1$, then we say that u is a weak solution of problem (1.2), if $u \in E$ and satisfies

$$\int_{\mathbb{R}^N} \nabla u \nabla \varphi \, dx = \int_{\mathbb{R}^N} f(u) \varphi \, dx + \int_{\Gamma_1} g(u) \varphi \, dx'$$

for any $\varphi \in W$.

Our first result concerns the nonexistence of positive solution for nonlinear-Neumann mixed boundary value problem, i.e., problem (1.1). Our main result is the following

Theorem 1.1. *Let $u \in E$ be a bounded nonnegative solution of problem (1.1), where $f, g : [0, +\infty) \rightarrow [0, +\infty)$ are continuous functions with the properties*

- (i) $f(t), g(t)$ are nondecreasing in $(0, +\infty)$.
 - (ii) $h(t) = \frac{f(t)}{t^{\frac{N+2}{N-2}}}, k(t) = \frac{g(t)}{t^{\frac{N}{N-2}}}$ are nonincreasing in $(0, +\infty)$.
- Then $u \equiv c$ with $f(c) = g(c) = 0$.*

Next, we study the mixed nonlinear-Dirichlet mixed boundary value problem (1.2). Our second conclusion is the following

Theorem 1.2. *Let $u \in E$ be a bounded nonnegative solution of problem (1.2), where $f, g : [0, +\infty) \rightarrow [0, +\infty)$ are continuous functions with the properties*

- (i) $f(t), g(t)$ are nondecreasing in $(0, +\infty)$.
 - (ii) $h(t) = \frac{f(t)}{t^{\frac{N+2}{N-2}}}, k(t) = \frac{g(t)}{t^{\frac{N}{N-2}}}$ are nonincreasing in $(0, +\infty)$.
- Then $u \equiv 0$.*

Remark 1.3. By the Doubling Lemma in [17], the boundedness assumptions in Theorem 1.1 and Theorem 1.2 can be dropped.

Here we should emphasize that we only need that the nonlinearities are continuous in the above theorems, we don't need they are Lipschitz continuous. Since no Lipschitz assumption was made on the nonlinearities, the usual maximum principle does not work. We resorted to some integral inequality, which was first introduced by S.Terracini in [22][23] and then was widely used in [6][12][13] and [14] respectively. By the same spirit of this, X.Yu studied the nonlinear Liouville type theorems for other equations in [24][25][26][27]

and [28]. The main idea of this paper is the same as the above works, we use integral inequality to substitute the usual maximum principle.

The rest of this paper is devoted the proof of Theorem 1.1 and Theorem 1.2. We divide the proof of these theorems into the following steps. First, we show that the nonnegative solutions of problems (1.1) and (1.2) are nondecreasing as x_1 decreases. Next, we show that the nonnegative solutions either depend only on x_1 and x_N or are regular at infinity. Finally, if u depends only on x_1 and x_N , then the limit function $w(x_N) = \lim_{x_1 \rightarrow -\infty} u(x_1, x_N)$ exists and $w(x_N)$ satisfies a proper equation and it can only be the trivial solution under the assumptions in Theorem 1.1 and Theorem 1.2. On the other hand, if the solution is regular at infinity, then we get a contradiction directly from the monotonicity of u in x_1 direction. In the rest of this paper, we denote C by a positive constant, which may vary from line to line.

2. Proof of Theorem 1.1

The key step is to prove the monotonicity of the nonnegative solutions for problem (1.1) in the x_1 direction. We use the moving plane method to prove our result. The first step of moving plane is to show that this procedure can be started at some point. Since we don't know the decay behaviors of u , it seems difficult to use this method directly on u . So we make use of the Kelvin transformation v of u . More precisely, for any $\mu \in \mathbb{R}$, we denote $p_\mu = (\mu, 0, \dots, 0) \in \partial\mathbb{R}_+^N$ and define the Kelvin transformation v_μ of u at p_μ as

$$v_\mu(x) = \frac{1}{|x - p_\mu|^{N-2}} u\left(\frac{x - p_\mu}{|x - p_\mu|^2} + p_\mu\right). \quad (2.1)$$

Then we deduce from the definition of Kelvin transformation that $v_\mu(x)$ decays at the rate of $|x - p_\mu|^{2-N}$ as $|x| \rightarrow \infty$. In particular, we have

$$v_\mu(x) \in L^{2^*} \cap L^\infty(\mathbb{R}_+^N \setminus B_r(p_\mu)) \quad (2.2)$$

for any $r > 0$, where $2^* = \frac{2N}{N-2}$ is the usual Sobolev critical exponent. Moreover, a direct calculation shows that v_μ satisfies the following equation

$$\begin{cases} -\Delta v_\mu = \frac{f(|x-p_\mu|^{N-2}v_\mu(x))}{[|x-p_\mu|^{N-2}v_\mu(x)]^{\frac{N+2}{N-2}}} v_\mu(x)^{\frac{N+2}{N-2}}, & x \in \mathbb{R}_+^N, \\ \frac{\partial v_\mu}{\partial \nu} = \frac{g(|x-p_\mu|^{N-2}v_\mu(x))}{[|x-p_\mu|^{N-2}v_\mu(x)]^{\frac{N}{N-2}}} v_\mu(x)^{\frac{N}{N-2}}, & x \in \{x \in \partial\mathbb{R}_+^N : \mu - \frac{1}{\mu} < x_1 < \mu\}, \\ \frac{\partial v_\mu}{\partial \nu} = 0, & x \in \{x \in \partial\mathbb{R}_+^N : x_1 > \mu \text{ or } x_1 < \mu - \frac{1}{\mu}\} \end{cases} \quad (2.3)$$

for $\mu \geq 0$, where we interpret $\frac{1}{\mu} = +\infty$ for $\mu = 0$. However, for $\mu < 0$, v_μ satisfies

$$\begin{cases} -\Delta v_\mu = \frac{f(|x-p_\mu|^{N-2}v_\mu(x))}{[|x-p_\mu|^{N-2}v_\mu(x)]^{\frac{N+2}{N-2}}} v_\mu(x)^{\frac{N+2}{N-2}}, & x \in \mathbb{R}_+^N, \\ \frac{\partial v_\mu}{\partial \nu} = \frac{g(|x-p_\mu|^{N-2}v_\mu(x))}{[|x-p_\mu|^{N-2}v_\mu(x)]^{\frac{N}{N-2}}} v_\mu(x)^{\frac{N}{N-2}}, & x \in \{x \in \partial\mathbb{R}_+^N : x_1 < \mu \text{ or } x_1 > \mu - \frac{1}{\mu}\}, \\ \frac{\partial v_\mu}{\partial \nu} = 0, & x \in \{x \in \partial\mathbb{R}_+^N : \mu < x_1 < \mu - \frac{1}{\mu}\}. \end{cases} \quad (2.4)$$

Moreover, we infer from the definitions of h and k that v_μ satisfies

$$\begin{cases} -\Delta v_\mu = h(|x-p_\mu|^{N-2}v_\mu(x))v_\mu(x)^{\frac{N+2}{N-2}}, & x \in \mathbb{R}_+^N, \\ \frac{\partial v_\mu}{\partial \nu} = k(|x-p_\mu|^{N-2}v_\mu(x))v_\mu(x)^{\frac{N}{N-2}}, & x \in \{x \in \partial\mathbb{R}_+^N : \mu - \frac{1}{\mu} < x_1 < \mu\}, \\ \frac{\partial v_\mu}{\partial \nu} = 0, & x \in \{x \in \partial\mathbb{R}_+^N : x_1 > \mu \text{ or } x_1 < \mu - \frac{1}{\mu}\} \end{cases} \quad (2.5)$$

for $\mu \geq 0$ and v_μ satisfies

$$\begin{cases} -\Delta v_\mu = h(|x-p_\mu|^{N-2}v_\mu(x))v_\mu(x)^{\frac{N+2}{N-2}}, & x \in \mathbb{R}_+^N, \\ \frac{\partial v_\mu}{\partial \nu} = k(|x-p_\mu|^{N-2}v_\mu(x))v_\mu(x)^{\frac{N}{N-2}}, & x \in \{x \in \partial\mathbb{R}_+^N : x_1 < \mu \text{ or } x_1 > \mu - \frac{1}{\mu}\}, \\ \frac{\partial v_\mu}{\partial \nu} = 0, & x \in \{x \in \partial\mathbb{R}_+^N : \mu < x_1 < \mu - \frac{1}{\mu}\} \end{cases} \quad (2.6)$$

for $\mu < 0$.

We first study the case $\mu > 0$. We will show that the moving plane procedure can be carried out from ∞ to $\lambda = \mu$. For this purpose, we define

$$\Sigma_\lambda = \{x \in \mathbb{R}_+^N : x_1 > \lambda\}, \quad T_\lambda = \{x \in \mathbb{R}_+^N : x_1 = \lambda\}$$

and

$$\partial\Sigma_\lambda^1 = \{x \in \partial\Sigma_\lambda : x_N = 0\}.$$

Moreover, for any $x \in \Sigma_\lambda$, the reflection of x with respect to T_λ is

$$x^\lambda = \{2\lambda - x_1, x_2, \dots, x_N\}.$$

In the following, we denote by $u^\lambda(x) = u(x^\lambda)$ and $p_\mu^\lambda = \{2\lambda - \mu, 0, \dots, 0\}$. With the above notations, we have the following key lemma.

Lemma 2.1. *For any fixed $\lambda > \mu \geq 0$, the functions v_μ and $(v_\mu - v_\mu^\lambda)^+$ belong to $L^{2^*}(\Sigma_\lambda) \cap L^\infty(\Sigma_\lambda)$ with $2^* = \frac{2N}{N-2}$. Further more, if we denote $A_\mu^\lambda = \{x \in \Sigma_\lambda | v_\mu > v_\mu^\lambda\}$ and $B_\mu^\lambda = \{x \in \partial\Sigma_\lambda^1 | v_\mu(x) > v_\mu^\lambda(x)\}$, then there exists $C_\lambda > 0$, which is nonincreasing in λ , such that*

$$\int_{\Sigma_\lambda} |\nabla(v_\mu - v_\mu^\lambda)^+|^2 dx \leq C_\lambda \left(\int_{A_\mu^\lambda} \frac{1}{|x - p_\mu|^{2N}} dx \right)^{\frac{2}{N}} \cdot \left(\int_{\Sigma_\lambda} |\nabla(v_\mu - v_\mu^\lambda)^+|^2 dx \right). \quad (2.7)$$

Proof. Since $\lambda > \mu$, there exists $r > 0$ such that $\Sigma_\lambda \subset \mathbb{R}_+^N \setminus B_r(p_\mu)$. Hence, we deduce from equation (2.2) and the decay behavior of v_μ that

$$v_\mu, (v_\mu - v_\mu^\lambda)^+ \in L^{2^*}(\Sigma_\lambda) \cap L^\infty(\Sigma_\lambda).$$

Now we choose a cut off function $\eta = \eta_\varepsilon \in C(\overline{\mathbb{R}_+^N}, [0, 1])$ such that

$$\eta(x) = \begin{cases} 1 & \text{for } 2\varepsilon \leq |x - p_\mu^\lambda| \leq \frac{1}{\varepsilon}, \\ 0 & \text{for } |x - p_\mu^\lambda| < \varepsilon \text{ or } |x - p_\mu^\lambda| > \frac{2}{\varepsilon}, \end{cases}$$

$|\nabla \eta| \leq \frac{2}{\varepsilon}$ for $\varepsilon < |x - p_\mu^\lambda| < 2\varepsilon$ and $|\nabla \eta| \leq 2\varepsilon$ for $\frac{1}{\varepsilon} < |x - p_\mu^\lambda| < \frac{2}{\varepsilon}$. Moreover, if we define $\varphi = \varphi_\varepsilon = \eta_\varepsilon^2(v_\mu - v_\mu^\lambda)^+$ and $\psi = \psi_\varepsilon = \eta_\varepsilon(v_\mu - v_\mu^\lambda)^+$, then a direct calculation shows that

$$|\nabla \psi|^2 = \nabla(v_\mu - v_\mu^\lambda) \nabla \varphi + [(v_\mu - v_\mu^\lambda)^+]^2 |\nabla \eta|^2.$$

On the other hand, it is easy to see that equation

$$-\Delta(v_\mu(x) - v_\mu^\lambda(x)) = h(|x - p_\mu|^{N-2} v_\mu(x)) v_\mu(x)^{\frac{N+2}{N-2}} - h(|x^\lambda - p_\mu|^{N-2} v_\mu(x^\lambda)) v_\mu(x^\lambda)^{\frac{N+2}{N-2}} \quad (2.8)$$

holds in $\Sigma_\lambda \setminus \{p_\mu^\lambda\}$. So if we multiply equation (2.8) by φ , then we get

$$\begin{aligned} & \int_{\Sigma_\lambda \cap \{2\varepsilon \leq |x - p_\mu^\lambda| \leq \frac{1}{\varepsilon}\}} |\nabla(v_\mu - v_\mu^\lambda)^+|^2 dx \\ & \leq \int_{\Sigma_\lambda} |\nabla \psi|^2 dx \\ & = \int_{\Sigma_\lambda} \nabla(v_\mu - v_\mu^\lambda) \nabla \varphi dx + \int_{\Sigma_\lambda} [(v_\mu - v_\mu^\lambda)^+]^2 |\nabla \eta_\varepsilon|^2 dx \\ & = \int_{A_\mu^\lambda} -\Delta(v_\mu - v_\mu^\lambda) \varphi dx + \int_{\partial \Sigma_\lambda^1} \frac{\partial(v_\mu - v_\mu^\lambda)}{\partial \nu} \varphi dx' + I_\varepsilon \\ & \leq \int_{A_\mu^\lambda} [h(|x - p_\mu|^{N-2} v_\mu(x)) v_\mu^{\frac{N+2}{N-2}} - h(|x^\lambda - p_\mu|^{N-2} v_\mu^\lambda(x^\lambda)) v_\mu^{\frac{N+2}{N-2}}] \varphi dx + I_\varepsilon, \end{aligned} \quad (2.9)$$

where $I_\varepsilon = \int_{\Sigma_\lambda} [(v - v^\lambda)^+]^2 |\nabla \eta_\varepsilon|^2 dx$, and the last inequality follows from that $\frac{\partial(v - v^\lambda)}{\partial \nu} \leq 0$ on $\partial \Sigma_\lambda^1$. Since h is nonincreasing, the above equation implies

$$\begin{aligned} & \int_{\Sigma_\lambda \cap \{2\varepsilon \leq |x - p_\mu^\lambda| \leq \frac{1}{\varepsilon}\}} |\nabla(v_\mu - v_\mu^\lambda)^+|^2 dx \\ & \leq \int_{A_\mu^\lambda} h(|x - p_\mu|^{N-2} v_\mu(x)) [v_\mu^{\frac{N+2}{N-2}} - (v_\mu^\lambda)^{\frac{N+2}{N-2}}] \varphi dx + I_\varepsilon \\ & \leq C \int_{A_\mu^\lambda} h(|x - p_\mu|^{N-2} v_\mu(x)) v_\mu^{\frac{4}{N-2}} [v_\mu - v_\mu^\lambda] \varphi dx + I_\varepsilon. \end{aligned} \quad (2.10)$$

Moreover, since $\Sigma_{\lambda'} \subset \Sigma_\lambda$ for $\lambda' > \lambda$ and $|x - p_\mu|^{N-2}v_\mu(x)$ is bounded in Σ_λ , then there exists a constant C_λ , which is nonincreasing in λ , such that

$$\begin{aligned} & \int_{\Sigma_\lambda \cap \{2\varepsilon \leq |\xi - p^\lambda| \leq \frac{1}{\varepsilon}\}} |\nabla(v_\mu - v_\mu^\lambda)^+|^2 dx \\ & \leq C_\lambda \int_{A_\mu^\lambda} v_\mu^{\frac{4}{N-2}} [v_\mu - v_\mu^\lambda]^+ \varphi dx + I_\varepsilon. \end{aligned} \quad (2.11)$$

On the other hand, we deduce from the decay property of v_μ that

$$\begin{aligned} & \int_{\Sigma_\lambda \cap \{2\varepsilon \leq |\xi - p^\lambda| \leq \frac{1}{\varepsilon}\}} |\nabla(v_\mu - v_\mu^\lambda)^+|^2 dx \\ & \leq C_\lambda \int_{A_\mu^\lambda} \frac{1}{|x - p_\mu|^4} [(v_\mu - v_\mu^\lambda)^+]^2 \eta^2 dx + I_\varepsilon \\ & \leq C_\lambda \left(\int_{A_\mu^\lambda} \frac{1}{|x - p_\mu|^{2N}} dx \right)^{\frac{2}{N}} \left(\int_{\Sigma_\lambda} [(v_\mu - v_\mu^\lambda)^+]^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} + I_\varepsilon. \end{aligned} \quad (2.12)$$

We claim that $I_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. In fact, if we denote $D_\varepsilon^1 = \{x \in \Sigma_\lambda | \varepsilon < |x - p_\mu^\lambda| < 2\varepsilon\}$ and $D_\varepsilon^2 = \{x \in \Sigma_\lambda | \frac{1}{\varepsilon} < |x - p_\mu^\lambda| < \frac{2}{\varepsilon}\}$, then we get

$$\int_{D_\varepsilon^1} |\nabla \eta|^N dx \leq C \frac{1}{\varepsilon^N} \cdot \varepsilon^N = C.$$

Similarly, we also have

$$\int_{D_\varepsilon^2} |\nabla \eta|^N dx \leq C \varepsilon^N \cdot \frac{1}{\varepsilon^N} = C.$$

Hence, it follows from the Holder inequality that

$$I_\varepsilon \leq \left(\int_{D_\varepsilon^1 \cup D_\varepsilon^2} [(v_\mu - v_\mu^\lambda)^+]^{2^*} dx \right)^{\frac{2}{2^*}} \cdot \left(\int_{\Sigma} |\nabla \eta|^N dx \right)^{\frac{2}{N}} \rightarrow 0$$

as $\varepsilon \rightarrow 0$ since $(v - v^\lambda)^+ \in L^{2^*}(\Sigma_\lambda)$.

Finally, letting $\varepsilon \rightarrow 0$ in equation (2.12) and using the dominated convergence theorem, Sobolev inequality, we get

$$\int_{\Sigma_\lambda} |\nabla(v_\mu - v_\mu^\lambda)^+|^2 dx \leq C_\lambda \left(\int_{A_\mu^\lambda} \frac{1}{|x - p_\mu|^{2N}} dx \right)^{\frac{2}{N}} \int_{\Sigma_\lambda} |\nabla(v_\mu - v_\mu^\lambda)^+|^2 dx, \quad (2.13)$$

which completes the proof of this lemma. \square

Before we continue the proof of Theorem 1.1, we give some comments on Lemma 2.1. Since we don't require the nonlinear terms f and g are Lipschitz continuous, the usual maximum principle does not work. Thanks to equation (2.7), it can play the same role as the maximum principle. In fact, if we can prove

$$C_\lambda \left(\int_{A_\mu^\lambda} \frac{1}{|x - p_\mu|^{2N}} dx \right)^{\frac{2}{N}} < 1,$$

then we get $v_\mu \leq v_\mu^\lambda$, the same conclusion as the maximum principle implies.

The next lemma shows that we can start the moving plane from some place.

Lemma 2.2. *Under assumptions of Theorem 1.1, there exists $\lambda_0 > \mu$, such that for all $\lambda \geq \lambda_0$, we have $v_\mu \leq v_\mu^\lambda$ in Σ_λ .*

Proof. The conclusion of this lemma is a direct corollary of Lemma 2.1. In fact, by the decay speed of v_μ , we can choose λ_0 large enough such that

$$C_\lambda \left(\int_{\Sigma_\lambda} \frac{1}{|x - p_\mu|^{2N}} dx \right)^{\frac{2}{N}} < \frac{1}{2}$$

for all $\lambda \geq \lambda_0$, then equation (2.7) implies that

$$\int_{\Sigma_\lambda} |\nabla(v_\mu - v_\mu^\lambda)^+|^2 dx = 0.$$

The assertion follows. \square

Now we can move the plane from the right to the left such that $v_\mu \leq v_\mu^\lambda$ in Σ_λ and suppose this process stops at some λ_1 . More precisely, we define

$$\lambda_1 = \inf\{\lambda | v_\mu \leq v_\mu^\lambda \text{ in } \Sigma_\lambda\}, \quad (2.14)$$

then we have the following

Lemma 2.3. *If $u \not\equiv 0$, then $\lambda_1 \leq \mu$.*

Proof. We prove the conclusion by contradiction. Suppose on the contrary that $\lambda_1 > \mu$, then we claim that $v_\mu \equiv v_\mu^{\lambda_1}$.

We prove the claim by contradiction. Suppose $v_\mu \not\equiv v_\mu^{\lambda_1}$, then we show that the plane can be moved to the left a little. That is, we will show that there exists $\delta > 0$, such that $v_\mu(x) \leq v_\mu^\lambda(x)$ in Σ_λ for all $\lambda \in [\lambda_1 - \delta, \lambda_1]$. This contradicts the choice of λ_1 .

To prove this claim, we first infer from the continuity that $v_\mu(x) \leq v_\mu^{\lambda_1}(x)$. Moreover, we have

$$\begin{aligned} h(|x - p_\mu|^{N-2} v_\mu) v_\mu^{\frac{N+2}{N-2}} &= \frac{f(|x - p_\mu|^{N-2} v_\mu)}{|x - p_\mu|^{N+2}} \\ &\leq \frac{f(|x - p_\mu|^{N-2} v_\mu^{\lambda_1})}{|x - p_\mu|^{N+2}} \\ &= \frac{f(|x - p_\mu|^{N-2} v_\mu^{\lambda_1})}{[|x - p_\mu|^{N-2} v_\mu^{\lambda_1}]^{\frac{N+2}{N-2}}} (v_\mu^{\lambda_1})^{\frac{N+2}{N-2}} \\ &\leq \frac{f(|x^{\lambda_1} - p_\mu|^{N-2} v_\mu^{\lambda_1})}{[|x^{\lambda_1} - p_\mu|^{N-2} v_\mu^{\lambda_1}]^{\frac{N+2}{N-2}}} (v_\mu^{\lambda_1})^{\frac{N+2}{N-2}} \\ &= h(|x^{\lambda_1} - p_\mu|^{N-2} v_\mu^{\lambda_1}) (v_\mu^{\lambda_1})^{\frac{N+2}{N-2}}, \end{aligned}$$

where the first inequality holds since f is nondecreasing, the second inequality is a consequence of (ii) in Theorem 1.1.

This equation implies

$$-\Delta v_\mu \leq -\Delta v_\mu^{\lambda_1},$$

then we infer from the maximum principle that $v_\mu < v_\mu^{\lambda_1}$ in Σ_{λ_1} since $v_\mu \not\equiv v_\mu^{\lambda_1}$. Moreover, since $\frac{1}{|x-p_\mu|^{2N}} \chi_{A_\mu^\lambda} \rightarrow 0$ a.e. as $\lambda \rightarrow \lambda_1$ and $\frac{1}{|x-p_\mu|^{2N}} \chi_{A_\mu^\lambda} \leq \frac{1}{|x-p_\mu|^{2N}} \chi_{A_\mu^{\lambda_1-\delta}}$ for $\lambda \in [\lambda_1 - \delta, \lambda_1]$ and some $\delta > 0$, then the dominated convergence theorem implies

$$\int_{A_\mu^\lambda} \frac{1}{|x-p_\mu|^{2N}} dx \rightarrow 0$$

as $\lambda \rightarrow \lambda_1$. That is, there exists $\delta > 0$, such that

$$C_\lambda \left(\int_{A_\mu^\lambda} \frac{1}{|x-p_\mu|^{2N}} dx \right)^{\frac{2}{N}} < \frac{1}{2}$$

for all $\lambda \in [\lambda_1 - \delta, \lambda_1]$. Then we infer from Lemma 2.1 that $v_\mu \leq v_\mu^\lambda$ for all $\lambda \in [\lambda_1 - \delta, \lambda_1]$. This contradicts the definition of λ_1 . So we prove the claim.

Next, we show that $v_\mu \equiv v_\mu^{\lambda_1}$ implies $v_\mu \equiv 0$ and hence $u \equiv 0$. In fact, if $v_\mu \equiv v_\mu^{\lambda_1}$, then we deduce from the Neumann boundary value condition and the nonlinear boundary value condition that $v_\mu(x) = 0$ for some $x \in \partial \mathbb{R}_+^N$, this contradicts the Hopf Theorem unless $v_\mu \equiv 0$. \square

The above two lemmas implies the following

Corollary 2.4. *For any $\mu \geq 0$, we have $v_\mu(x) \leq v_\mu(x^\mu)$ for $x \in \Sigma_\mu$.*

Next, we consider the case $\mu < 0$. In this case, it seems difficult to start the moving plane method as $\mu > 0$ because of the boundary value condition of v_μ . Instead of choosing μ arbitrary for $\mu \geq 0$, we move μ from 0 to $-\infty$ gradually. We first need some integral inequality similar to equation (2.7) to substitute the maximum principle. In this case, we need to evaluate the term $\int_{\partial \Sigma_\lambda^1} \frac{\partial(v-v^\lambda)}{\partial \nu} \varphi dx'$ in equation (2.9) more carefully since it is not nonpositive any longer. Similar to Lemma 2.1, we have the following result.

Lemma 2.5. *Suppose that $\mu < 0$, then for any fixed $\mu < \lambda \leq \mu - \frac{1}{2\mu}$, the functions v_μ and $(v_\mu - v_\mu^\lambda)^+$ belong to $L^{2^*}(\Sigma_\lambda) \cap L^\infty(\Sigma_\lambda)$ with $2^* = \frac{2N}{N-2}$. Further more, if we denote $A_\mu^\lambda = \{x \in \Sigma_\lambda | v_\mu > v_\mu^\lambda\}$ and $B_\mu^\lambda = \{x \in \partial \Sigma_\lambda^1 | v_\mu(x) > v_\mu^\lambda(x)\}$, then there exists $C_\lambda > 0$, which is nonincreasing in λ , such that*

$$\begin{aligned} & \int_{\Sigma_\lambda} |\nabla(v_\mu - v_\mu^\lambda)^+|^2 dx \\ & \leq C_\lambda \left[\left(\int_{A_\mu^\lambda} \frac{1}{|x-p_\mu|^{2N}} dx \right)^{\frac{2}{N}} + \left(\int_{B_\mu^\lambda} \frac{1}{|x-p_\mu|^{2(N-1)}} dx' \right)^{\frac{1}{N-1}} \right] \cdot \left(\int_{\Sigma_\lambda} |\nabla(v_\mu - v_\mu^\lambda)^+|^2 dx \right). \end{aligned} \quad (2.15)$$

Proof. The proof of this lemma is similar to the proof of Lemma 2.1. We sketch it. We denote η, φ, ψ as the proof of Lemma 2.1, then we have the following inequality similar to

equation (2.9). It reads

$$\begin{aligned}
& \int_{\Sigma_\lambda \cap \{2\varepsilon \leq |\xi - p^\lambda| \leq \frac{1}{\varepsilon}\}} |\nabla(v_\mu - v_\mu^\lambda)^+|^2 dx \\
& \leq \int_{A_\mu^\lambda} -\Delta(v_\mu - v_\mu^\lambda) \varphi dx + \int_{\partial \Sigma_\lambda^1} \frac{\partial(v_\mu - v_\mu^\lambda)}{\partial \nu} \varphi dx' + I_\varepsilon \\
& \leq \int_{A_\mu^\lambda} [h(|x - p_\mu|^{N-2} v_\mu(x)) v_\mu^{\frac{N+2}{N-2}} - h(|x^\lambda - p_\mu|^{N-2} v_\mu^\lambda) (v_\mu^\lambda)^{\frac{N+2}{N-2}}] \varphi dx \\
& \quad + \int_{B_\mu^\lambda} [k(|x - p_\mu|^{N-2} v_\mu(x)) v_\mu^{\frac{N}{N-2}} - k(|x^\lambda - p_\mu|^{N-2} v_\mu^\lambda) (v_\mu^\lambda)^{\frac{N}{N-2}}] \varphi dx' + I_\varepsilon,
\end{aligned} \tag{2.16}$$

where $I_\varepsilon = \int_{\Sigma_\lambda} [(v_\mu - v_\mu^\lambda)^+]^2 |\nabla \eta_\varepsilon|^2 dx$. Since h and k are nonincreasing, the above equation implies

$$\begin{aligned}
& \int_{\Sigma_\lambda \cap \{2\varepsilon \leq |\xi - p^\lambda| \leq \frac{1}{\varepsilon}\}} |\nabla(v_\mu - v_\mu^\lambda)^+|^2 dx \\
& \leq \int_{A_\mu^\lambda} h(|x - p_\mu|^{N-2} v_\mu(x)) [v_\mu^{\frac{N+2}{N-2}} - (v_\mu^\lambda)^{\frac{N+2}{N-2}}] \varphi dx \\
& \quad + \int_{B_\mu^\lambda} k(|x - p_\mu|^{N-2} v_\mu(x)) [v_\mu^{\frac{N}{N-2}} - (v_\mu^\lambda)^{\frac{N}{N-2}}] \varphi dx' + I_\varepsilon \\
& \leq C \int_{A_\mu^\lambda} h(|x - p_\mu|^{N-2} v_\mu(x)) v_\mu^{\frac{4}{N-2}} [v_\mu - v_\mu^\lambda] \varphi dx \\
& \quad + C \int_{B_\mu^\lambda} k(|x - p_\mu|^{N-2} v_\mu(x)) v_\mu^{\frac{2}{N-2}} [v_\mu(x) - v_\mu(x^\lambda)] \varphi dx' + I_\varepsilon. \\
& \leq C_\lambda \int_{A_\mu^\lambda} v_\mu^{\frac{4}{N-2}} [v_\mu - v_\mu^\lambda]^+ \varphi dx + C_\lambda \int_{B_\mu^\lambda} v_\mu^{\frac{2}{N-2}} [v_\mu(x) - v_\mu(x^\lambda)] \varphi dx' + I_\varepsilon \\
& \leq C_\lambda \left(\int_{A_\mu^\lambda} \frac{1}{|x - p_\mu|^{2N}} dx \right)^{\frac{2}{N}} \left(\int_{\Sigma_\lambda} [(v_\mu - v_\mu^\lambda)^+]^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} \\
& \quad + C_\lambda \left(\int_{B_\mu^\lambda} \frac{1}{|x - p_\mu|^{2(N-1)}} dx' \right)^{\frac{1}{N-1}} \left(\int_{\partial \Sigma_\lambda^1} [(v_\mu - v_\mu^\lambda)^+]^{\frac{2(N-1)}{N-2}} dx' \right)^{\frac{N-2}{N-1}} + I_\varepsilon \\
& \leq C_\lambda \left[\left(\int_{A_\mu^\lambda} \frac{1}{|x - p_\mu|^{2N}} dx \right)^{\frac{2}{N}} + \left(\int_{B_\mu^\lambda} \frac{1}{|x - p_\mu|^{2(N-1)}} dx' \right)^{\frac{1}{N-1}} \right] \cdot \left(\int_{\Sigma_\lambda} |\nabla(v_\mu - v_\mu^\lambda)^+|^2 dx \right) + I_\varepsilon.
\end{aligned} \tag{2.17}$$

Letting $\varepsilon \rightarrow 0$ in the above equation and using the dominated convergence theorem, Sobolev inequality and Sobolev trace inequality, then we get equation (2.15), which completes the proof of this lemma. \square

With the above preparations, we can start the moving plane procedure for $\mu < 0$ now. However, in the case $\mu < 0$, it seems difficult to move the plane T_λ because of the boundary value conditions of v_μ . Instead of choosing μ arbitrary for $\mu \geq 0$, we move the plane μ from 0 to $-\infty$ step by step. In the first step, we will show that the parameter μ can be moved from 0 to $\mu_1 = -\frac{\sqrt{2}}{2}$. This procedure is composed of the following two lemmas. First, we show that this procedure can be started for μ small enough. More precisely, we have the following result.

Lemma 2.6. *There exists $\mu_0 < 0$, such that for all $\mu_0 \leq \mu \leq 0$, we have $v_\mu \leq v_\mu^0$ for all $x \in \Sigma_0$.*

Proof. By Corollary 2.4, we know that $v_0 \leq v_0^0$ for all $x \in \Sigma_0$, hence the functions $\frac{1}{|x-p_\mu|^{2N}} \chi_{A_\mu^0} \rightarrow 0$ and $\frac{1}{|x-p_\mu|^{2(N-1)}} \chi_{B_\mu^0} \rightarrow 0$ a.e. as $\mu \rightarrow 0^-$. We deduce from the dominate convergence theorem that there exists $\mu_0 < 0$ small enough, such that for all $\mu_0 \leq \mu \leq 0$, the following inequality holds

$$C_\lambda \left[\left(\int_{A_\mu^0} \frac{1}{|x-p_\mu|^{2N}} dx \right)^{\frac{2}{N}} + \left(\int_{B_\mu^0} \frac{1}{|x-p_\mu|^{2(N-1)}} dx' \right)^{\frac{1}{N-1}} \right] \leq \frac{1}{2}.$$

Then we infer from Lemma 2.5 that $\int_{A_\mu^0} |\nabla(v_\mu - v_\mu^0)^+| dx = 0$, that is $v_\mu \leq v_\mu^0$ in Σ_0 . □

Next, we will show that the plane can be moved down to $-\frac{\sqrt{2}}{2}$. More precisely, we define

$$\mu_1 = \inf\{\mu < 0 : v_\mu \leq v_\mu^0 \text{ in } \Sigma_0\},$$

then we have

Lemma 2.7. *If $u \not\equiv 0$, then we must have $\mu_1 \leq -\frac{\sqrt{2}}{2}$.*

Proof. We prove the conclusion by contradiction. Suppose on the contrary that $\mu_1 > -\frac{\sqrt{2}}{2}$, then we claim that $v_{\mu_1} \equiv v_{\mu_1}^0$.

We prove the claim by contradiction. Suppose $v_{\mu_1} \not\equiv v_{\mu_1}^0$, then we show that the plane can be moved to the left a little. That is, we will show that there exists $\delta > 0$, such that $v_\mu(x) \leq v_\mu^0(x)$ in Σ_0 for all $\mu \in [\mu_1 - \delta, \mu_1]$. This contradicts the choice of μ_1 .

To prove this, we first infer from the continuity that $v_{\mu_1}(x) \leq v_{\mu_1}^0(x)$ for $x \in \Sigma_0$. Moreover, we have

$$\begin{aligned}
h(|x - p_{\mu_1}|^{N-2} v_{\mu_1}) v_{\mu_1}^{\frac{N+2}{N-2}} &= \frac{f(|x - p_{\mu_1}|^{N-2} v_{\mu_1})}{|x - p_{\mu_1}|^{N+2}} \\
&\leq \frac{f(|x - p_{\mu_1}|^{N-2} v_{\mu_1}^0)}{|x - p_{\mu_1}|^{N+2}} \\
&= \frac{f(|x - p_{\mu_1}|^{N-2} v_{\mu_1}^0)}{[|x - p_{\mu_1}|^{N-2} v_{\mu_1}^0]^{\frac{N+2}{N-2}}} (v_{\mu_1}^0)^{\frac{N+2}{N-2}} \\
&\leq \frac{f(|x^0 - p_{\mu_1}|^{N-2} v_{\mu_1}^0)}{[|x^0 - p_{\mu_1}|^{N-2} v_{\mu_1}^0]^{\frac{N+2}{N-2}}} (v_{\mu_1}^0)^{\frac{N+2}{N-2}} \\
&= h(|x^0 - p_{\mu_1}|^{N-2} v_{\mu_1}^0) (v_{\mu_1}^0)^{\frac{N+2}{N-2}},
\end{aligned}$$

where the first inequality holds since f is nondecreasing, the second inequality is a consequence of (ii) in Theorem 1.1.

This equation implies

$$-\Delta v_{\mu_1} \leq -\Delta v_{\mu_1}^0,$$

then we infer from the maximum principle that $v_{\mu_1} < v_{\mu_1}^0$ in Σ_0 since $v_{\mu_1} \not\equiv v_{\mu_1}^0$. Since $\frac{1}{|x-p_\mu|^{2N}} \chi_{A_\mu^0} \rightarrow 0$, $\frac{1}{|x-p_\mu|^{2(N-1)}} \chi_{B_\mu^0} \rightarrow 0$ a.e. as $\mu \rightarrow \mu_1$, moreover, as $\frac{1}{|x-p_\mu|^{2N}} \chi_{A_\mu^0} \leq \frac{1}{|x-p_\mu|^{2N}} \chi_{A_{\mu_1-\delta}^0}$ and $\frac{1}{|x-p_\mu|^{2(N-1)}} \chi_{B_\mu^0} \leq \frac{1}{|x-p_\mu|^{2(N-1)}} \chi_{B_{\mu_1-\delta}^0}$ for $\mu \in [\mu_1 - \delta, \mu_1]$ and some $\delta > 0$, then the dominated convergence theorem implies

$$\int_{A_\mu^0} \frac{1}{|x - p_\mu|^{2N}} dx + \int_{B_\mu^0} \frac{1}{|x - p_\mu|^{2(N-1)}} dx' \rightarrow 0$$

as $\mu \rightarrow \mu_1$. That is, there exists $\delta > 0$, such that

$$C_\lambda \left[\left(\int_{A_\mu^0} \frac{1}{|x - p_\mu|^{2N}} dx \right)^{\frac{2}{N}} + \left(\int_{B_\mu^0} \frac{1}{|x - p_\mu|^{2(N-1)}} dx' \right)^{\frac{1}{N-1}} \right] < \frac{1}{2}$$

for all $\mu \in [\mu_1 - \delta, \mu_1]$. Then we infer from Lemma 2.5 that $v_\mu \leq v_\mu^0$ for all $\mu \in [\mu_1 - \delta, \mu_1]$. This contradicts the definition of μ_1 . This proves the claim.

Similar to Lemma 2.3, we can show that $v_{\mu_1}(x) \equiv v_{\mu_1}^0$ implies $v_{\mu_1} \equiv 0$. In fact, if $v_{\mu_1} \equiv v_{\mu_1}^0$, then we deduce from the Neumann boundary value condition and the nonlinear boundary value condition that $v_{\mu_1}(x) = 0$ for some $x \in \partial \mathbb{R}_+^N$, this contradicts the Hopf Theorem unless $v_{\mu_1} \equiv 0$. If $v_{\mu_1} \equiv 0$, then the proof of Theorem 1.1 is complete. So if $u \not\equiv 0$, then we must have $\mu_1 \leq -\frac{\sqrt{2}}{2}$. \square

Now, let $\mu \in [\mu_1, 0)$ fixed, then the same procedure as above shows that we move can the plane T_λ from $\lambda = 0$ to $\lambda = \mu$, in particular, we have $v_\mu(x) \leq v_\mu(x^\mu)$ for $x \in \Sigma_\mu$ and $\mu \in [\mu_1, 0)$. Based on the information that $v_{\mu_1}(x) \leq v_{\mu_1}(x^{\mu_1})$ for $x \in \Sigma_{\mu_1}$, we can move the plane μ from $\mu_1 = -\frac{\sqrt{2}}{2}$ to $\mu_2 = -\frac{\sqrt{2}+3}{4}$, then for $\mu \in [\mu_2, \mu_1)$, we can move the plane T_λ from μ_1 to μ . In particular, we have $v_\mu(x) \leq v_\mu(x^\mu)$ for $x \in \Sigma_\mu$ and $\mu \in [\mu_2, \mu_1)$.

Continuing the above procedure, in the $(k+1)$ th step, we can show that the parameter μ can be moved from μ_k to $\mu_{k+1} = \frac{\mu_k - \sqrt{\mu_k^2 + 2}}{2}$. A direct calculation shows that $\mu_k \rightarrow -\infty$ as $k \rightarrow \infty$. Hence, we have proved

Proposition 2.8. *For any $\mu < 0$, we have $v_\mu(x) \leq v_\mu(x^\mu)$ for $x \in \Sigma_\mu$.*

Proposition 2.9. *For fixed $(x_2, x_3, \dots, x_N) \in \mathbb{R}^{N-1}$, $u(x_1, x_2, x_3, \dots, x_N)$ is nondecreasing as x_1 decreases.*

Proof. By Corollary 2.4 and Proposition 2.8, we have $v_\mu \leq v_\mu^\mu$ for any $\mu \in \mathbb{R}$. In particular, we have

$$\frac{1}{|x - p_\mu|^{N-2}} v(p_\mu + \frac{x - p_\mu}{|x - p_\mu|^2}) \leq \frac{1}{|x_\mu - p_\mu|^{N-2}} v(p_\mu + \frac{x^\mu - p_\mu}{|x^\mu - p_\mu|^2}).$$

Since $|x - p_\mu| = |x^\mu - p_\mu|$ and $x_1^\mu - \mu = \mu - x_1$, then we have

$$u(x_1, x_2, x_3, \dots, x_N) \leq u(2\mu - x_1, x_2, x_3, \dots, x_N)$$

for $x_1 \geq \mu$. Because μ is arbitrary, then we conclude that u is nondecreasing as x_1 decreases. \square

Proof of Theorem 1.1: To prove Theorem 1.1, we first start the moving plane method in the x_2 direction. For this purpose, we define the Kelvin transformation v_μ of u at $p_\mu = (0, \mu, 0, \dots, 0)$ as

$$v_\mu(x) = \frac{1}{|x - p_\mu|^{N-2}} u(p_\mu + \frac{x - p_\mu}{|x - p_\mu|^2}).$$

Moreover, we define $\Sigma_\lambda = \{x \in \mathbb{R}_+^N \mid x_2 > \lambda\}$, $T_\lambda = \{x \in \mathbb{R}_+^N \mid x_2 = \lambda\}$ and x^λ be the reflection of x with respect to T_λ as before.

If we denote $v_\mu^\lambda = v_\mu(x^\lambda)$, then by the same reason as Lemma 2.1 and Lemma 2.5, we have the following inequality

$$\begin{aligned} & \int_{\Sigma_\lambda} |\nabla(v - v^\lambda)^+|^2 dx \\ & \leq C_\lambda \left[\left(\int_{A_\mu^\lambda} \frac{1}{|x - p_\mu|^{2N}} dx \right)^{\frac{2}{N}} + \left(\int_{B_\mu^\lambda} \frac{1}{|x - p_\mu|^{2(N-1)}} dx' \right)^{\frac{1}{N-1}} \right] \cdot \left(\int_{\Sigma_\lambda} |\nabla(v - v^\lambda)^+|^2 dx \right), \end{aligned} \quad (2.18)$$

where C_λ is a nonincreasing constant, $A_\mu^\lambda = \{x \in \Sigma_\lambda \mid v_\mu(x) > v_\mu^\lambda\}$ and

$B_\mu^\lambda = \{x \in \partial\Sigma_\lambda \mid v_\mu(x) > v_\mu^\lambda, x_N = 0\}$. With the above inequality, we can start the moving plane method for λ large enough. Now, move the plane T_λ from the right to the left and suppose the process stops at some λ_1 . If $\lambda_1 > \mu$, then we can prove that v_μ is symmetric with respect to T_{λ_1} as before. This means that v_μ is regular at p_μ . In particular, we have $u \in L^{\frac{2N}{N-2}}(\mathbb{R}_+^N)$. Otherwise, we must have $\lambda_1 \leq \mu$, which implies $v_\mu \leq v_\mu^\mu$. At the same time, we can start moving plane method from the left. Similarly, if this process stops at some $\lambda_1 < \mu$, then $u \in L^{\frac{2N}{N-2}}(\mathbb{R}_+^N)$, otherwise, we have $\lambda_1 = \mu$ and $v_\mu^\mu \leq v_\mu$.

We have proved the following fact: if the moving plane procedure stops at some $\lambda_1 \neq \mu$, then $u \in L^{\frac{2N}{N-2}}(\mathbb{R}_+^N)$, otherwise, we must have $v_\mu \equiv v_\mu^\mu$ and hence $u \equiv u^\mu$.

With the above preparations, we can prove Theorem 1.1 now. We distinguish two cases.

Case 1: During the moving plane method process in the x_2, \dots, x_{N-1} directions, there exists some μ and some direction, such that $\lambda_1 \neq \mu$, i.e., the moving plane method stops at some $\lambda_1 \neq \mu$. Then we conclude that u is regular at infinity and hence $u \in L^{\frac{2N}{N-2}}(\mathbb{R}_+^N)$. This is impossible unless $u \equiv 0$ since u is nondecreasing as x_1 decreases by Proposition 2.9.

Case 2: For all μ and all directions x_2, \dots, x_{N-1} , the moving plane method stops at $\lambda_1 = \mu$. In this case, the function u is independent of x_2, \dots, x_{N-1} since μ is arbitrary. So we have $u = u(x_1, x_N)$. Moreover, Proposition 2.9 implies that $u(x_1, x_N)$ is nondecreasing as x_1 decreases. On the other hand, since u is bounded, then the limit $w(x_N) = \lim_{x_1 \rightarrow -\infty} u(x_1, x_N)$ exists and $w(x_N)$ satisfies

$$\begin{cases} w''(x_N) = -f(w) & \text{for } x_N > 0, \\ \frac{\partial w}{\partial x_N} = -g(u) & \text{for } x_N = 0. \end{cases} \quad (2.19)$$

If $w \not\equiv c$ with $f(c) = g(c) = 0$, then the first equation implies that w is a concave function, while the second equation implies $w'(0) < 0$. Hence there exists $r_0 > 0$ such that $w(x_N) < 0$ for $x_N > r_0$, this contradicts that u is a nonnegative solution to problem (1.1).

Finally, since $w \equiv c$ with $f(c) = g(c) = 0$ and f, g are nondecreasing, then equation (1.1) turns to be

$$\begin{cases} -\Delta u = 0 & \text{in } \mathbb{R}_+^N, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial \mathbb{R}_+^N. \end{cases}$$

Since u is bounded, then the classical Liouville theorem implies $u \equiv c$ with $f(c) = g(c) = 0$. This finishes the proof of Theorem 1.1. \square

3. Proof of Theorem 1.2

The sprit of the proof of Theorem 1.2 is the same as the proof of Theorem 1.1. We only give the difference between the two problems.

As before, for any $\mu \in \mathbb{R}$, we define the Kelvin transformation v_μ of u at $p_\mu = (\mu, 0, \dots, 0)$ as

$$v_\mu(x) = \frac{1}{|x - p_\mu|^{N-2}} u(p_\mu + \frac{x - p_\mu}{|x - p_\mu|^2}),$$

then a direct calculations shows that v_μ satisfies

$$\begin{cases} -\Delta v_\mu = \frac{f(|x - p_\mu|^{N-2} v_\mu(x))}{[|x - p_\mu|^{N-2} v_\mu(x)]^{\frac{N+2}{N-2}}} v_\mu(x)^{\frac{N+2}{N-2}}, & x \in \mathbb{R}_+^N, \\ \frac{\partial v_\mu}{\partial \nu} = \frac{g(|x - p_\mu|^{N-2} v_\mu(x))}{[|x - p_\mu|^{N-2} v_\mu(x)]^{\frac{N}{N-2}}} v_\mu(x)^{\frac{N}{N-2}}, & x \in \{x \in \partial \mathbb{R}_+^N : \mu - \frac{1}{\mu} < x_1 < \mu\}, \\ v_\mu = 0, & x \in \{x \in \partial \mathbb{R}_+^N : x_1 > \mu \text{ or } x_1 < \mu - \frac{1}{\mu}\} \end{cases} \quad (3.1)$$

for $\mu \geq 0$, where we interpret $\frac{1}{\mu} = +\infty$ for $\mu = 0$. However, for $\mu < 0$, v_μ satisfies

$$\begin{cases} -\Delta v_\mu = \frac{f(|x-p_\mu|^{N-2}v_\mu(x))}{[|x-p_\mu|^{N-2}v_\mu(x)]^{\frac{N+2}{N-2}}} v_\mu(x)^{\frac{N+2}{N-2}}, & x \in \mathbb{R}_+^N, \\ \frac{\partial v_\mu}{\partial \nu} = \frac{g(|x-p_\mu|^{N-2}v_\mu(x))}{[|x-p_\mu|^{N-2}v_\mu(x)]^{\frac{N}{N-2}}} v_\mu(x)^{\frac{N}{N-2}}, & x \in \{x \in \partial\mathbb{R}_+^N : x_1 < \mu \text{ or } x_1 > \mu - \frac{1}{\mu}\}, \\ v_\mu = 0, & x \in \{x \in \partial\mathbb{R}_+^N : \mu < x_1 < \mu - \frac{1}{\mu}\}. \end{cases} \quad (3.2)$$

Moreover, we infer from the definitions of h and k that v_μ satisfies

$$\begin{cases} -\Delta v_\mu = h(|x-p_\mu|^{N-2}v_\mu(x))v_\mu(x)^{\frac{N+2}{N-2}}, & x \in \mathbb{R}_+^N, \\ \frac{\partial v_\mu}{\partial \nu} = k(|x-p_\mu|^{N-2}v_\mu(x))v_\mu(x)^{\frac{N}{N-2}}, & x \in \{x \in \partial\mathbb{R}_+^N : \mu - \frac{1}{\mu} < x_1 < \mu\}, \\ v_\mu = 0, & x \in \{x \in \partial\mathbb{R}_+^N : x_1 > \mu \text{ or } x_1 < \mu - \frac{1}{\mu}\} \end{cases} \quad (3.3)$$

for $\mu \geq 0$ and v_μ satisfies

$$\begin{cases} -\Delta v_\mu = h(|x-p_\mu|^{N-2}v_\mu(x))v_\mu(x)^{\frac{N+2}{N-2}}, & x \in \mathbb{R}_+^N, \\ \frac{\partial v_\mu}{\partial \nu} = k(|x-p_\mu|^{N-2}v_\mu(x))v_\mu(x)^{\frac{N}{N-2}}, & x \in \{x \in \partial\mathbb{R}_+^N : x_1 < \mu \text{ or } x_1 > \mu - \frac{1}{\mu}\}, \\ v_\mu = 0, & x \in \{x \in \partial\mathbb{R}_+^N : \mu < x_1 < \mu - \frac{1}{\mu}\} \end{cases} \quad (3.4)$$

for $\mu < 0$.

As before, we first start the moving plane method in the x_1 direction for $\mu \geq 0$. For problem (3.4), we have the same inequality as Lemma 1.2.

Lemma 3.1. *For any fixed $\lambda > \mu \geq 0$, the functions v_μ and $(v_\mu - v_\mu^\lambda)^+$ belong to $L^{2^*}(\Sigma_\lambda) \cap L^\infty(\Sigma_\lambda)$ with $2^* = \frac{2N}{N-2}$. Further more, if we denote $A_\mu^\lambda = \{x \in \Sigma_\lambda | v_\mu > v_\mu^\lambda\}$ and $B_\mu^\lambda = \{x \in \partial\Sigma_\lambda^1 | v_\mu(x) > v_\mu^\lambda(x)\}$, then there exists $C_\lambda > 0$, which is nonincreasing in λ , such that*

$$\int_{\Sigma_\lambda} |\nabla(v_\mu - v_\mu^\lambda)^+|^2 dx \leq C_\lambda \left(\int_{A_\mu^\lambda} \frac{1}{|x-p_\mu|^{2N}} dx \right)^{\frac{2}{N}} \cdot \left(\int_{\Sigma_\lambda} |\nabla(v_\mu - v_\mu^\lambda)^+|^2 dx \right). \quad (3.5)$$

Proof. We observe from the boundary value condition of v_μ that

$$\frac{\partial(v_\mu - v_\mu^\lambda)}{\partial \nu} \varphi(x) = 0$$

for any $x \in \partial\Sigma_\lambda^1$. Then the rest of the proof is the same as the proof of Lemma 2.1. \square

With the above inequality, we can start the moving plane procedure now. Similar to the proof of Theorem 1.1, we have

Proposition 3.2. *For any $\mu \geq 0$, we have $v_\mu(x) \leq v_\mu^\mu(x)$ for $x \in \Sigma_\mu$.*

Next, we consider the case $\mu < 0$. Similar to the result for problem (1.1), we have the following result.

Lemma 3.3. *Suppose that $\mu < 0$, then for any fixed $\mu < \lambda < \mu - \frac{1}{2\mu}$, the functions v_μ and $(v_\mu - v_\mu^\lambda)^+$ belong to $L^{2^*}(\Sigma_\lambda) \cap L^\infty(\Sigma_\lambda)$ with $2^* = \frac{2N}{N-2}$. Further more, if we denote $A_\mu^\lambda = \{x \in \Sigma_\lambda | v_\mu > v_\mu^\lambda\}$ and $B_\mu^\lambda = \{x \in \partial\Sigma_\lambda^1 | v_\mu(x) > v_\mu^\lambda(x)\}$, then there exists $C_\lambda > 0$, which is nonincreasing in λ , such that*

$$\begin{aligned} & \int_{\Sigma_\lambda} |\nabla(v_\mu - v_\mu^\lambda)^+|^2 dx \\ & \leq C_\lambda \left[\left(\int_{A_\mu^\lambda} \frac{1}{|x - p_\mu|^{2N}} dx \right)^{\frac{2}{N}} + \left(\int_{B_\mu^\lambda} \frac{1}{|x - p_\mu|^{2(N-1)}} dx' \right)^{\frac{1}{N-1}} \right] \cdot \left(\int_{\Sigma_\lambda} |\nabla(v_\mu - v_\mu^\lambda)^+|^2 dx \right). \end{aligned} \quad (3.6)$$

Proof. We only need to note that

$$\int_{\partial\Sigma_\lambda} \frac{\partial(v_\mu - v_\mu^\lambda)}{\partial\nu} \varphi(x) dx' \leq \int_{B_\mu^\lambda} [k(|x - p_\mu|^{N-2} v_\mu(x)) v_\mu^{\frac{N}{N-2}} - k(|x^\lambda - p_\mu|^{N-2} v_\mu^\lambda(x)) (v_\mu^\lambda)^{\frac{N}{N-2}}] \varphi dx'.$$

The rest of the proof is the same the proof of Lemma 2.5, we omit it. \square

With the above inequality, we can start the moving plane procedure for $\mu < 0$. First, we can move μ from $\mu_0 = 0$ to $\mu_1 = -\frac{\sqrt{2}}{2}$, then for $\mu \in [\mu_1, 0)$, we can move the plane T_λ from $\lambda = 0$ to $\lambda = \mu$. That is, we have $v_\mu(x) \leq v_\mu(x^\mu)$ for any $\mu \in [\mu_1, 0)$ and $x \in \Sigma_\mu$. Next, we can move the plane μ from μ_1 to $\mu_2 = -\frac{3+\sqrt{2}}{4}$. Similarly, for any $\mu \in [\mu_2, \mu_1)$, we can move the plane T_λ from $\lambda = \mu_1$ to $\lambda = \mu$. That is, we have $v_\mu(x) \leq v_\mu(x^\mu)$ for any $\mu \in [\mu_2, \mu_1)$ and $x \in \Sigma_\mu$. Continue the above procedure, in the $(k+1)$ th step, the plane μ can be moved from μ_k from $\mu_{k+1} = \frac{\mu_k - \sqrt{\mu_k^2 + 2}}{2}$ and T_λ can be moved from μ_k to $\mu \in [\mu_{k+1}, \mu_k)$. A direct calculation shows that $\mu_k \rightarrow -\infty$ as $k \rightarrow \infty$. So we have proved the following result.

Proposition 3.4. *For any $\mu < 0$, we have $v_\mu(x) \leq v_\mu^\mu(x)$ for $x \in \Sigma_\mu$.*

As a direct consequence of Proposition 3.2 and Proposition 3.4, we have the following result.

Proposition 3.5. *For fixed $(x_2, x_3, \dots, x_N) \in \mathbb{R}^{N-1}$, $u(x_1, x_2, x_3, \dots, x_N)$ is nondecreasing as x_1 decreases.*

With the above results, we are in the position of proving Theorem 1.2 now.

Proof of Theorem 1.2. As the proof of Theorem 1.1, we first start the moving plane method in the x_2 direction. For this purpose, we define the Kelvin transformation v_μ of u at $p_\mu = (0, \mu, 0, \dots, 0)$ as

$$v_\mu(x) = \frac{1}{|x - p_\mu|^{N-2}} u\left(p_\mu + \frac{x - p_\mu}{|x - p_\mu|^2}\right).$$

and $\Sigma_\lambda, T_\lambda, x^\lambda$ as before.

If we denote $v_\mu^\lambda = v_\mu(x^\lambda)$, then by the same reason as Lemma 3.1 and Lemma 3.3, we have the following inequality

$$\begin{aligned} & \int_{\Sigma_\lambda} |\nabla(v - v^\lambda)^+|^2 dx \\ & \leq C_\lambda \left[\left(\int_{A_\mu^\lambda} \frac{1}{|x - p_\mu|^{2N}} dx \right)^{\frac{2}{N}} + \left(\int_{B_\mu^\lambda} \frac{1}{|x - p_\mu|^{2(N-1)}} dx' \right)^{\frac{1}{N-1}} \right] \cdot \left(\int_{\Sigma_\lambda} |\nabla(v - v^\lambda)^+|^2 dx \right) \end{aligned} \quad (3.7)$$

with C_λ, A_λ and B_λ as before. With this inequality, we can start the moving plane method for λ large enough. Now, moving the plane T_λ from the right to the left and suppose that the process stops at some λ_1 . If $\lambda_1 > \mu$, then we can prove that v_μ is symmetric with respect to T_{λ_1} as before. This means that v_μ is regular at p_μ . In particular, we have $u \in L^{\frac{2N}{N-2}}(\mathbb{R}_+^N)$. Otherwise, we must have $\lambda_1 \leq \mu$, which implies $v_\mu \leq v_\mu^\mu$. At the same time, we can start the moving plane method from the left. Similarly, if this process stops at some $\lambda_1 < \mu$, then $u \in L^{\frac{2N}{N-2}}(\mathbb{R}_+^N)$, otherwise, we have $\lambda_1 = \mu$ and $v_\mu^\mu \leq v_\mu$.

So during the process of moving plane in the x_2, \dots, x_{N-1} directions, two cases may occur.

Case 1: There exists some direction and some μ , such that $\lambda_1 \neq \mu$, i.e., the moving plane method stops at some $\lambda_1 \neq \mu$. Then we conclude that u is regular at infinity and hence $u \in L^{\frac{2N}{N-2}}(\mathbb{R}_+^N)$. This is impossible unless $u \equiv 0$ since u is nondecreasing as x_1 decrease by Proposition 3.5.

Case 2: For all μ and all directions x_2, \dots, x_{N-1} , the moving plane method stops at $\lambda_1 = \mu$. In this case, the function u is independent of x_2, \dots, x_{N-1} since μ is arbitrary. So we have $u = u(x_1, x_N)$. Moreover, Proposition 3.5 implies that $u(x_1, x_N)$ is nondecreasing as x_1 decreases. On the other hand, since u is bounded, then the limit function $w(x_N) = \lim_{x_1 \rightarrow -\infty} u(x_1, x_N)$ exists and $w(x_N)$ satisfies

$$\begin{cases} w''(x_N) = -f(w) & \text{for } x_N > 0, \\ \frac{\partial w}{\partial x_N} = -g(u) & \text{for } x_N = 0. \end{cases} \quad (3.8)$$

If $w \not\equiv c$ with $f(c) = g(c) = 0$, then the first equation implies that w is a concave function, while the second equation implies $w'(0) < 0$. Hence there exists $r_0 > 0$ such that $w(x_N) < 0$ for $x_N > r_0$, this contradicts that u is a nonnegative solution to problem (1.2). So we have $w \equiv c$ and the Dirichlet boundary value condition implies $u \equiv 0$. This finishes the proof of Theorem 1.2. □

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